Emergence of the Born Rule from Discrete Node Dynamics

Information Metric, Effective Hamiltonians, and Statistical Limits

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Abstract

We propose a discrete, node-based framework in which quantum mechanics emerges as a continuum approximation of underlying update dynamics. Fundamental entities are nodes related by causal update relations, with no probabilistic postulate assumed at the microscopic level. Transition amplitudes are assigned as minimal two-dimensional internal degrees of freedom, while observations correspond to coarse-grained node densities. We show that the Born rule is not a fundamental axiom but an inevitable quadratic statistical form arising from phase elimination, σ-additivity, and non-contextuality in the continuum limit. The Schrödinger equation appears as an effective generator of near-identity node updates, and probability measures are uniquely fixed by information-geometric consistency. This framework reinterprets quantum theory as a shadow of discrete node dynamics rather than a primary description of reality.

1 Introduction

The Born rule,

$$P(x) = |\psi(x)|^2, \tag{1}$$

is usually postulated as a fundamental axiom of quantum mechanics. Its conceptual status has long remained obscure: probability is introduced at a level where the theory otherwise exhibits deterministic, unitary evolution.

In this work, we demonstrate that the Born rule need not be postulated. Instead, it emerges as a statistical approximation when a discrete, non-probabilistic node dynamics is represented in a continuous form. Probability appears only after coarse-graining over node updates and eliminating non-observable phase information.

Our approach differs from standard interpretations by treating the wave function not as an ontological object, but as a collective descriptor of node amplitudes. Quantum mechanics is thereby reinterpreted as an effective theory, analogous to thermodynamics, rather than a fundamental description.

2 Minimal Postulates of Node Theory

2.1 Discrete Node Structure

Let the world be described by a countable set of nodes:

$$\mathcal{N} = \{n_i\}. \tag{2}$$

Nodes are not embedded in a pre-existing continuous spacetime. Instead, they are related by a partial order representing causal update relations. Time is not continuous but indexed by discrete update steps $k \in \mathbb{Z}$.

2.2 Node Transition Amplitudes

A transition from node n_i to node n_j carries an amplitude

$$a_{ij} \in \mathbb{R}^2 \simeq \mathbb{C}. \tag{3}$$

The use of complex numbers is purely representational, encoding the minimal two-dimensional internal degree of freedom required for interference. No assumption of fundamentally complex-valued reality is made.

A node configuration at update step k is described by

$$\Phi(k) = \{(n_i, a_i(k))\}. \tag{4}$$

At this microscopic level, no probability is defined.

2.3 Observation as Coarse-Graining

An observation corresponds to coarse-graining over a region $R \subset \mathcal{N}$. Phase information of individual amplitudes is unobservable; only aggregated quantities over many node updates are empirically accessible.

3 Emergence of the Wave Function

Introduce a coarse-grained cell cell(x) in a continuous label space. The effective wave function is defined as

$$\psi(x) := \sum_{n_i \in \mathrm{cell}(x)} a_i. \tag{5}$$

Key properties are:

- $\psi(x)$ is not fundamental.
- It is a collective variable summarizing many node amplitudes.
- It exists only after coarse-graining.

Thus, the wave function represents a quasi-stationary configuration of node densities rather than an elementary physical entity.

4 Why a Quadratic Measure Appears

4.1 Observable Quantities

Observation eliminates phase information. Any observable must therefore be invariant under the global phase transformation

$$\psi \mapsto e^{i\theta}\psi. \tag{6}$$

Linear quantities such as $\sum_i a_i$ are not invariant and hence unobservable. The simplest non-trivial invariant quantity is quadratic.

4.2 Empirical Frequency

Let N_R be the number of node update events in a region R and N the total number of events. The empirical frequency is

$$P(R) = \frac{N_R}{N}. \tag{7}$$

At the node level, this frequency is proportional to

$$P(R) \propto \sum_{n_i \in R} |a_i|^2. \tag{8}$$

4.3 Continuum Limit

In the continuum approximation,

$$P(x) \propto |\psi(x)|^2. \tag{9}$$

This quadratic form arises not from assumption but from necessity:

- phase information is unobservable,
- additivity must hold for disjoint regions,
- positivity and non-contextuality must be preserved.

The quadratic measure is therefore the unique consistent statistical description.

5 Status of the Born Rule

We thus identify

$$P(x) = |\psi(x)|^2 \tag{10}$$

as:

- not a fundamental law of reality,
- not an intrinsic property of the wave function,
- but an emergent statistical approximation resulting from discrete node structure, phase elimination, and large-number limits.

The Born rule is therefore analogous to thermodynamic laws arising from microscopic dynamics.

6 Effective Hamiltonian from Node Updates

6.1 Discrete Update Rule

Let $\phi_i(k)$ denote the amplitude at node i at update step k. The most general linear local update rule is

$$\phi_i(k+1) = \sum_j U_{ij},\ \phi_j(k), \tag{11}$$

where U preserves normalization:

$$\sum_i |\phi_i|^2 = \text{const.} \tag{12}$$

6.2 Near-Identity Limit

Assume that updates are small:

$$U = I - i\epsilon H + O(\epsilon^2). \tag{13}$$

Then

$$\phi(k+1) - \phi(k) = -i\epsilon H,\ \phi(k). \tag{14}$$

In the continuum limit,

$$i\frac{\partial \phi}{\partial t} = H\phi. \tag{15}$$

The Schrödinger equation thus appears as an effective description of node updates. Self-adjointness of H follows directly from norm preservation, not from an independent postulate.

7 Information Metric from Node Density

Define the node density

$$\rho(x) = |\psi(x)|^2. \tag{16}$$

The distinguishability between two nearby densities ρ and $\rho + \delta\rho$ is measured by the Fisher information metric

$$\mathrm{d}s^2 = \int \mathrm{d}x,\ \frac{(\delta\rho)^2}{\rho}. \tag{17}$$

This metric is uniquely determined by positivity, locality, and reparameterization invariance. Probability and geometry are therefore linked by information-theoretic necessity.

8 Uniqueness of the Quadratic Measure (Gleason-Type Result)

Theorem. Let $\mathcal{H}$ be a Hilbert space representing node amplitudes. Let

$$P : \{\text{orthogonal projections on } \mathcal{H}\} \to [0, 1] \tag{18}$$

satisfy σ-additivity, phase invariance, and non-contextuality. Then there exists a unique density operator ρ such that

$$P(\Pi) = \mathrm{Tr}(\rho,\ \Pi). \tag{19}$$

For pure states,

$$P(\psi) = \langle\psi|\psi\rangle. \tag{20}$$

Thus, the quadratic rule is structurally inevitable rather than optional.

9 Conclusion

Within the node framework presented here:

- quantum mechanics is not fundamental,
- probability is not primitive,
- the Born rule is an emergent statistical law.

Quantum theory appears as a continuum approximation of discrete node dynamics, just as thermodynamics emerges from molecular motion. The mystery of quantum probability is resolved not by interpretation, but by structural necessity.

References

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